Isotropy subgroups and Orbits :

$$\begin{aligned} G \times X &\xrightarrow{\Phi} X \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$ group action.

(I) $x \in X$, (fixed). Consider

$$G_x = \{ g \in G \mid g \cdot x = x \} \leq G \qquad (\text{check !})$$

It is called the isotropy subgroup of G at x.

Thus $X \longrightarrow$ ~~Subgro~~ set of subgroups of G

$$x \longmapsto G_x$$

(i) Let $\phi : G \longrightarrow \mathrm{Perm}(X)$ be the corresponding homo.

then $\mathrm{Ker}(\phi) = \bigcap_{x \in X} G_x \triangleleft G$.

Def: the action Φ is ~~called~~ called "faithful" if $\mathrm{ker}(\phi) = \{e\}$.

In other words, Φ is faithful $\Longleftrightarrow$ $\forall g \in G$, $\exists$ some $x \in X$

s.t $g \cdot x \neq x$.

(ii) $x' = g \cdot x$. Then $G_{x'} = g G_x g^{-1}$ (check)

Question: Assume for two pts $x, x' \in X$, $\exists g \in G$, s.t

$G_{x'} = g \cdot G_x \cdot g^{-1}$. Does it follow $x' = g \cdot x$?

Example:

(1) $G \times G \to G$ conjugation

$(g, x) \mapsto gxg^{-1}$

$G_x = N_x$ normalizer ;

$G \times X \to X$, X = set of subgroups/subsets of G

$(g, S) \mapsto gSg^{-1}$

$G_S = N_S$ normalizer

(2) $G \times G \to G$ (Left) translation.

$(g, x) \longmapsto g \cdot x$

$G_x = \{e\}$, $\forall x \in G$

(3) $V \cong k^n$, $GL(V) \times V \to V$

$(g, v) \longmapsto g \cdot v$

$v = 0$, $\Rightarrow G_v = GL(V)$

Fix an iso. $V \underset{\simeq}{\overset{\psi}{\to}} K^n$ (by choosing a K-basis of V).

$$0 \neq v \longmapsto e_1$$

Such that $\psi(v) = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Determine

$$G_{e_1} \overset{\Delta}{=} \left\{ A \in GL_n(k) \;\middle|\; A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ A = \left(\begin{array}{c|c} 1 & b \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array} \right) \;\middle|\; \begin{array}{l} B \in GL_{n-1}(k) \\ b \in K^{n-1} \end{array} \right\}$$

$$\left(\begin{array}{c|c} 1 & b_1 \\ \hline 0 & B_1 \end{array} \right) \cdot \left(\begin{array}{c|c} 1 & b_2 \\ \hline 0 & B_2 \end{array} \right) = \left(\begin{array}{c|c} 1 & b_2 + b_1 \cdot B_2 \\ \hline 0 & B_1 B_2 \end{array} \right)$$

Check: The above subgrp $G_{e_1} \cong$ the affine transformation group of K^{n-1}.

Note: $GL(V) \overset{\Psi}{\cong} GL_n(k)$

$$f \longmapsto \psi \circ f \circ \psi^{-1}$$

Check: $GL(V)_v = \Psi^{-1}(G_{e_1})$.

$\mathbb{P}(V) \triangleq$ the set of 1-dim linear subspaces in V.

Then
$$\begin{array}{ccc} GL(V) \times \mathbb{P}(V) & \longrightarrow & \mathbb{P}(V) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g, [v]) & \longmapsto & [gv], \end{array}$$

where $[v]$ means the 1-dim. ~~subspace~~ subspace spanned by $v \neq 0 \in V$.

Q: what is $GL(V)_{[v]}$?

In general, one defines, for $1 \leq r \leq \dim_k V - 1$

$$Gr(r, V) = \{[W] \mid \underset{\text{linear subspace}}{W \subset V}, \dim_k W = r\}$$

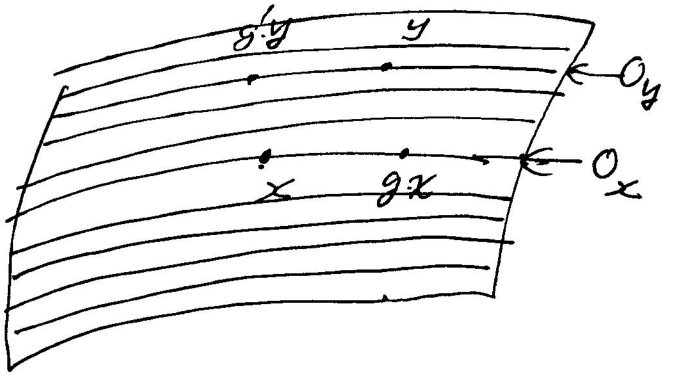
$$\begin{array}{ccc} GL(V) \times Gr(r, V) & \longrightarrow & Gr(r, V) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g, [W]) & \longmapsto & [g.W] \end{array}$$

Q: what is $GL(V)_{[W]}$?

(II) $x \in X$, (fixed). Consider

$$\mathcal{O}_x = \underset{G \cdot x}{\cancel{G \cdot \{x\}}} = \{g \cdot x \mid g \in G\} \subseteq X$$

It is called the orbit of x (under the G-action)



The most distinguished property of orbit is the following:

$\forall\, x, y \in X$, either

(*): $\quad O_x = O_y \quad$ or $\quad O_x \cap O_y = \phi$.

Pf. ~~A~~ssume $O_x \cap O_y \neq \phi$

ie $\exists\ z \in X$. and $g_1, g_2 \in G$, s.t

$$\left.\begin{array}{l} x = g_1 \cdot z \\ y = g_2 \cdot z \end{array}\right\} \Rightarrow x = (g_1 \cdot g_2^{-1}) \cdot y$$

$$\Rightarrow O_x = \underset{G \cdot x}{\cancel{G \cdot \{x\}}} = G \cdot (g_1 \cdot g_2^{-1}) \cdot y$$

$$= G \cdot y = O_y \qquad \#$$

Conclusion: $X = \coprod_{x \in S} O_x \qquad S \subseteq X$ subset.

Def (fixed pt & transitive action).

If $x \in X$, $O_x = \{x\}$. then $x \in X$ is called a fixed pt of

G-action (i.e. $g \cdot x = x$, $\forall g \in G$).

If $\exists x \in X$, $O_x = X$, then, the action is called "transitive".

Note; if the action is transitive, then $\forall y \in X$, $O_y = X$.

(i.e. there is only one orbit in X)

Prop: Fix any $x \in X$, then one has the natural map

$$\begin{array}{ccc} G/G_x & \longrightarrow & O_x \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ gG_x & \longmapsto & gx \end{array}$$

, which is a bijection.

Consequently, $|O_x| = [G : G_x)$.

Pf: Exercise.

#

Cor: If $|X| < +\infty$, then

$$\# \quad |X| = \sum_{x \in S} (G : G_x).$$

Cor (Class formula): Assume $|G| < +\infty$. Then

$$|G| = \sum_{x \in C} [G : \cancel{N}_x)$$

C = the set of conjugacy classes of G.

~~Note~~,

$$= |Z(G)| + \sum_{x \in C \setminus Z(G)} (G : N_x),$$

Example (modular curve)

$$G_{\mathbb{Z}} = SL_2(\mathbb{Z}) \leq G_{\mathbb{R}} = SL_2(\mathbb{R}), \quad X = \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{im}(z) > 0\}$$

Note $G_{\mathbb{R}} \times X \longrightarrow X$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \longmapsto \frac{az+b}{cz+d}$$

Check: $\operatorname{im} z > 0 \Rightarrow \operatorname{im}\left(\frac{az+b}{cz+d}\right) > 0$

(1) The action is transitive.

Solve: $\forall\, z = x+iy$, $y>0$, find $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,

s.t. $\dfrac{ai+b}{ci+d} = x+iy$

$$\Leftrightarrow \quad (*) \begin{cases} ad-bc = 1 \\ a = cx+dy \\ b = dx-cy \end{cases}$$

Take $c=0$.

$$(*) \Leftrightarrow \begin{cases} ad = 1 \\ a/d = y \\ b/d = x \end{cases}$$

Take $d = \frac{1}{\sqrt{y}}$, $a = \sqrt{y}$, $b = x/\sqrt{y}$, $c=0$.

(2) The isotropy subgp of $G_{\mathbb{R}}$ at i

$$= \left\{ \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in [0, 2\pi) \right\} \cong SO(2)$$

Indeed:

$$\frac{ai+b}{ci+d} = i \,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

$$\Longleftrightarrow \begin{cases} ad-bc = 1 \\ a = d \\ b = -c \end{cases}$$

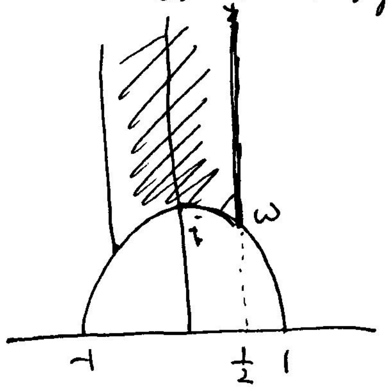
$\Longleftrightarrow$ $a = \cos\theta$, $b = \sin\theta$, $c = -\sin\theta$, $d = \cos\theta$ for some $\theta \in \mathbb{R}$.

(3) Since $G_{\mathbb{Z}} \leq G_{\mathbb{R}}$, $G_{\mathbb{Z}} \times \mathbb{H} \longrightarrow \mathbb{H}$ naturally by restricting the $G_{\mathbb{R}}$-action to the subgp $G_{\mathbb{Z}}$.

The orbit space

$$G_{\mathbb{Z}} \backslash \mathbb{H} = \text{the set of } G_{\mathbb{Z}}\text{-orbits}$$

is called the modular curve.



Lecture 6. Sylow's Theorem

Recall: (Lagrange)

$|G| = n, \quad H \leq G \Rightarrow |H| \mid n$

Question: $|G| = n$, $d \mid n$. Does there exist

$H \leq G$ with $|H| = d$?

We know this is true, if G is cyclic.

How about G abelian ?

Example: $|A_n| = \frac{1}{2} n!$

Then for $n \geq 5$, $4 \mid n!$

But $\nexists\, H \leq A_n$, s.t. $|H| = \frac{1}{4} n!$

(why ?)

Sylow answered the above question in a beautiful way:

Def: $|G| = n = p_1^{r_1} \cdots p_s^{r_s}$. Let $p_i \in \{p_1 \cdots p_s\}$.

A sylow p_i-subgroup of G is a subgroup of order $= p_i^{r_i}$.

Theorem (Sylow)

$|G| = n = p_1^{r_1} \cdots p_s^{r_s}$. ~~Then p=2k~~, for each $p \in \{p_1, \ldots, p_s\}$.

(I) sylow p-subgroup always exists.

(II) All sylow p-subgroups are conjugate

(III) The number of p-sylow subgps $\equiv 1 \mod p$.

Structure of p-groups:

A group G is called a p-group, if $|G| = p^r$ for some prime p.

Theorem: G is a ~~non-trivial~~ p-group. Then

(1) $Z(G) \neq \{e\}$

(2) G is solvable

(3) G admits a normal tower

$$G = G_0 \supset G_1 \supset \cdots \supset G_{n+1} = \{e\}$$

s.t $G_i / G_{i+1} \cong \mathbb{Z}/p\mathbb{Z}$, $0 \le i \le n$.

Corollary: $|G| = n$, Then $\forall\, p^r \mid n$, p prime

there exists a subgp $H \leq G$, with $|H| = p^r$.

We prove the second theorem first:

Proof: (1) By the class formula,

$$p^n = |G| = |Z(G)| + \sum_{x \in C \setminus Z(G)} [G : N_x]$$

Note, $\forall x \in C \setminus Z(G)$, $|C_x| \doteq \dfrac{|G|}{|N_x|} \geqslant 2$

$$\Rightarrow p \,\|\, |C_x|$$

$\Rightarrow$ $|Z(G)| = p^n - \sum_{x \in C \setminus Z(G)} [G : N_x]$ is also ~~divisible~~ divisible by p.

Since $e \in Z(G)$, $|Z(G)| \geq 2$.

$\Rightarrow Z(G) \neq \{e\}$.

(2) Do induction on $|G|$. and use

$$G \xrightarrow{\pi} G/Z(G)$$

(3) Note $Z(G)$ is abelian; ~~and is~~

Thus take any $x^{\cancel{x}e} \in Z(G)$, $|\langle x \rangle| = p^r$

If $r=1$, then we get an elt of order of p.

Otherwise, we replace x by $x^{p^{r-1}}$. So we can assume $\langle x\rangle \cong \mathbb{Z}/p\mathbb{Z}$.

Then we consider

$$G \xrightarrow{\pi} G/\langle x\rangle$$

and use induction on $|G|$.

#

Now we prove Sylow's theorem. The key idea is to use the group actions.

proof: (I). Do induction on $|G|$.

~~WLOG~~, We can assume each proper subgp $H \lneq G$, with the property. $p \mid [G:H]$. Otherwise, a sylow p-subgp of H is already a sylow p-subgp of G, which exists by induction.

Now. Consider again

$$|G| = |Z(G)| + \sum_{x \in C\setminus Z(G)} [G : N_x]$$

A similar argument to the proof of the above theorem

Shows that $Z(G) \neq \{e\}$; and $p \mid |Z(G)|$

Use the fact ~~that~~ see page 58 for a proof):

G : abelian, finite, $\forall p \mid |G|$, there exists an elt of order p.

We take $x \in Z(G)$, with $\mathrm{ord}(x) = p$.

Consider $\pi : G \longrightarrow G/\langle x \rangle$

By induction, $\exists$ sylow p-group $\bar{H} \leq G/\langle x \rangle$.

Then claim $\pi^{-1}(\bar{H}) \leq G$ is a sylow p group of G.

Indeed: $p^r \mid |G|$, $p^{r+1} \nmid |G|$. Then

$|\bar{H}| = p^{r-1}$, because $p^{r-1} \mid |G/\langle x \rangle|$, $p^r \nmid |G/\langle x \rangle|$.

However, it is clearly that

$$\pi^{-1}(\bar{H}) / \langle x \rangle \simeq \bar{H}.$$

we get $|\pi^{-1}(\bar{H})| = |\bar{H}| \cdot |\langle x \rangle| = p^r$.

#

Before proving (II) & (III), we show a simple Lemma.

Lemma: H is a p-group (i.e. $|H| = p^r$).

$H \times X \to X$ group action. Then:

(i) The number of fixed pts of $H \equiv |X|$ mod p

(ii) If H has exactly one fixed pt, then $|X| \equiv 1$ mod p

(iii) If $p \mid |X|$, then the number of fixed pts of H is $\equiv 0$ mod p.

$$F(X) \; \| \; \text{Fixed pts of } H = \{x \in X \,/\, |O_x| = 1\}$$

Pf: (i) Then

$$|X| = \sum_{x \in S} |O_x| = \#\text{ fixed pts of } X + \sum_{\substack{x \in S \\ x \notin F(X)}} [H : H_x]$$

But, $\forall x \in S \setminus F(X)$, $2 \le |O_x| \mid |H| = p^r$. $\Rightarrow p \mid |O_x|$

$\Rightarrow |X| \equiv \# F(X)$ mod p.

(ii), (iii) follows from (i).

#

Proof: X = set of all sylow p-gps in G

By (I), $|X| \neq \phi$.

Consider the natural conjugation-action

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g, [H]) & \longmapsto & [gHg^{-1}] \end{array}$$

$[H] \in X$ means $H \leq G$ is a sylow p-gp.

If $|X| = 1$, then we're done.

So we assume $|X| \geq 2$. in the following.

Note (iii) follows from (ii) :

Consider
$$\begin{array}{ccc} H \times X & \longrightarrow & X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (h, [H']) & \longmapsto & [hH'h^{-1}] \end{array}$$

Certainly, the H-action has $[H]$ as a fixed pt.

Claim: The H-action has a unique fixed pt. (that is $[H]$).

pf of claim: Suppose $[H']$ is ~~a~~ another fixed pt. under the H-action.

Then $\forall h \in H$, $hH'h^{-1} = H'$

$\iff H \leq N_{H'}$

But $H' \leq N_{H'}$, Thus, we get two sylow p-subgps

H and H' in $N_{H'}$.

By (iv), $\exists\, \tilde{h} \in N_{H'}$, s.t

$$\tilde{h} H' \tilde{h}^{-1} = H$$
$$\parallel$$
$$H'$$

Contradicts with $H \neq H'$.

Now we prove (ii). which says that the G-action on X is transitive, i.e.

We prove by contradiction. Then

$$X = \coprod_{[H] \in S} O_{[H]}, \qquad O_{[H]} = \{[gHg^{-1}] / g \in G\}$$

$\forall\, [H_1], [H_2] \in X$, Consider the action

$$\begin{array}{ccc} H_1 \times O_{[H_2]} & \longrightarrow & O_{[H_2]} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (h_1, [H]) & \longmapsto & [h_1 H h_1^{-1}] \end{array} \qquad (*)$$

Case (1): $\exists\, [H] \in X$, s.t $|O_{[H]}| \geq 2$.

Note: $\left.\begin{array}{l} |O_{[H]}| = |G : N_H| \\ H \leq N_H \not\leq G \end{array}\right\} \Rightarrow (|O_{[H]}|, p) = 1.$

Put $H_2 = H$ in $(*)$.

By Lemma (1), because $|O_{[H]}| \not\equiv 0 \mod p$,

we find a fixed pt under H_1-action.

We can assume ~~this fi.~~ $[H_2]$ is a fixed pt (otherwise, we replace H_2 by a suitable conjugation).

Then the H_1-action fixes $[H_2]$

$$\Longleftrightarrow \quad H_1 \leq \underset{\overset{\mathbin{\rotatebox{90}{$\geq$}}}{H_2}}{N_{H_2}} \lneq G$$

Do induction on $|G|$. We can assume

$$\exists\, \tilde{h} \in N_{H_2}, \text{ s.t } \underset{\overset{\shortparallel}{H_2}}{\tilde{h} H_2 \tilde{h}^{-1}} = H_1$$

This means exactly, $H_1 \in O_{[H_2]}$. $\forall [H_1] \in X$.

Contradiction!

Case (2), $\forall [H] \in X$, $|O_{[H]}| = 1$, i.e. $O_{[H]} = \{[H]\}$.

This means, $\forall H \leq G$, sylow p-subgp $H \trianglelefteq G$.

Then we take $H_1 \neq H_2$, two sylow p-subgp. Then $H_1 \cdot H_2$ is a p-subgp, ~~which is strictly~~ which $H_i \lneq H_1 H_2$. Contradiction! #